设函数 $f(x)$ 在 $[a,b]$ 上不恒为 0，且导数 $f'(x)$ 连续，$f(a)=f(b)=0$

证明：$\exists \xi \in [a,b]$，使得 $f'(\xi) = \dfrac{1}{(b-a)^2}\int_a^b f(x)\,dx$

令 $G(x) = \int_a^x f(t)\,dt$，$G'(x) = f(x)$，$G''(x) = f'(x)$，$G(a) = 0$

即证：$\exists \xi \in [a,b]$，使得 $G''(\xi) > \dfrac{1}{(b-a)^2} G(b)$

$f(b) = f(a) = 0 \Rightarrow G'(a) = G'(b) = 0$

① $G(b) \neq 0$ 时

$G(a)$ 在 b 处展开（因为要求 $G(b)$）

$$G(a) = G(b) + (a-b)G'(b) + \frac{(a-b)^2}{2} G''(\delta_1),\quad \delta_1 \in (a,b)$$

$$0 = G(b) + 0 + \frac{(a-b)^2}{2} G''(\delta_1)$$

$$\Rightarrow G''(\delta_1) = \frac{-2}{(a-b)^2} G(b)$$

$$\frac{-2}{(a-b)^2} < \frac{1}{(a-b)^2}$$

$G(b) < 0$ 时，有 $G''(\delta_1) = \dfrac{-2}{(a-b)^2} G(b) > \dfrac{1}{(a-b)^2} G(b)$

但 $G(b) > 0$ 时，$G''(\delta_1) < \dfrac{1}{(a-b)^2} G(b)$，结论不成立，找对偶情况

$G(b)$ 在 a 处展开

$$G(b) = G(a) + (b-a)G'(a) + \frac{(b-a)^2}{2} G''(\delta_2),\quad \delta_2 \in (a,b)$$

$$G(b) = \frac{(b-a)^2}{2} G''(\delta_2)$$

$$\Rightarrow G''(\delta_2) = \frac{2}{(b-a)^2} G(b)$$

$$\frac{2}{(b-a)^2} > \frac{1}{(b-a)^2}$$

$G(b) > 0$ 时，有 $G''(\delta_2) = \dfrac{2}{(a-b)^2} G(b) > \dfrac{1}{(a-b)^2} G(b)$

综上 $\begin{cases} G(b) > 0, & \xi = \delta_2 \\ G(b) > 0, & \xi = \delta_1 \end{cases}$ 因为只需 $\exists \xi \in (a,b)$ 即可（做两手准备）

② $G(b) = 0$ 时，即证 $\exists \xi \in (a,b)$ $f'(\xi) > 0$

$f(x)$ 不恒为 0 $\Rightarrow$ $\exists\, c \in [a,b]$, $f(c) \neq 0$

$f(a) = f(b) = 0 \Rightarrow c \in (a,b)$

不妨设 $f(c) > 0$

$\exists\, \xi \in (0, c)$, $f'(\xi) = \dfrac{f(c)-f(0)}{c-0} = \dfrac{f(c)}{c} > 0$

设函数 $f(x)$ 在 $[0,1]$ 上有连续的导函数，且 $\int_0^1 f(x)dx = 0$

证明：对 $\forall x \in (0,1)$，有 $\left|\int_0^x f(t)dt\right| \leq \frac{1}{8}\max\limits_{x\in[0,1]}|f'(x)|$

令 $G(x) = \int_0^x f(t)dt \qquad G'(x) = f(x) \qquad G''(x) = f'(x)$

即证 $|G(x)| \leq \frac{1}{8}\max\limits_{x\in[0,1]}|G''(x)|$, $\quad G(0) = G(1) = 0$

$G(0) = G(x) + G'(x)(0-x) + \dfrac{G''(\xi_1)}{2}(0-x)^2, \quad \xi_1 \in (0,x)$

$0 = G(x) - xG'(x) + \dfrac{G''(\xi_1)}{2}x^2$

$G'(x) = \frac{1}{x}G(x) + \dfrac{G''(\xi_1)}{2}x$

$G(1) = G(x) + G'(x)(1-x) + \dfrac{G''(\xi_2)}{2}(1-x)^2, \quad \xi_2 \in (x,1)$

$0 = G(x) + G'(x)(1-x) + \dfrac{G''(\xi_2)}{2}(1-x)^2$

$G'(x) = \dfrac{G(x)}{x-1} + \dfrac{G''(\xi_2)}{2}(x-1)$

$G'(x) = \dfrac{G(x)}{x} + \dfrac{G''(\xi_1)}{2}x = \dfrac{G(x)}{x-1} + \dfrac{G''(\xi_2)}{2}(x-1)$

$\left(\frac{1}{x} - \frac{1}{x-1}\right)G(x) = \dfrac{G''(\xi_2)}{2}(x-1) - \dfrac{G''(\xi_1)}{2}x$

$\dfrac{1}{x(1-x)}G(x) = \dfrac{G''(\xi_2)}{2}(x-1) - \dfrac{G''(\xi_1)}{2}x$

$G(x) = -\dfrac{G''(\xi_2)}{2}x(1-x)^2 - \dfrac{G''(\xi_1)}{2}x^2(1-x)$

$|G(x)| = \left|\dfrac{G''(\xi_2)}{2}x(1-x)^2 + \dfrac{G''(\xi_1)}{2}x^2(1-x)\right|$

$$\le M\cdot\frac{1}{2}x(1-x)(1-x+x)=M\frac{1}{2}x(1-x)\le\frac{1}{8}M$$

其中 $M=\max\limits_{x\in[0,1]}|f'(x)|$

设 $f(x)$ 在 $[0,1]$ 上有一阶连续导函数，且 $f(0)=f(1)=0$

求证：$\left|\int_0^1 f(x)dx\right|\le\frac{1}{4}\max\limits_{0\le x\le1}|f'(x)|$

令 $G(x)=\int_0^x f(t)dt$，$G(0)=0$

$G'(x)=f(x)$，$G''(x)=f'(x)$

$f(0)=f(1)\Rightarrow G'(0)=G'(1)$

即证：$|G(1)|\le\frac{1}{4}\max\limits_{0\le x\le1}|G''(x)|$

$G(1)=G(0)+G'(0)\cdot(1-0)+\frac{G''(\xi_1)}{2}(1-0)^2$，$\xi_1\in(0,1)$

$G(1)=\frac{G''(\xi_1)}{2}$

$G(0)=G(1)+G'(1)\cdot(0-1)+\frac{G''(\xi_2)}{2}(0-1)^2$，$\xi_2\in(0,1)$

$G(1)=-\frac{G''(\xi_2)}{2}$

$G(1)=\frac{1}{2}\left(\frac{G''(\xi_1)}{2}-\frac{G''(\xi_2)}{2}\right)$

$|G(1)|=\frac{1}{2}\left|\frac{G''(\xi_1)}{2}-\frac{G''(\xi_2)}{2}\right|\le\frac{1}{4}|G''(\xi_1)|+\frac{1}{4}|G''(\xi_2)|=\frac{1}{2}M$

其中 $M=\max\limits_{x\in[0,1]}|f'(x)|$，不得证

(其它尝试)，待定 $G(x)$ 在 0、1 处展开，解出可用的 x

$G(x)=G(0)+G'(0)\cdot x+\frac{G''(\xi_1)}{2}x^2$，$\xi_1\in(0,x)$

$G(x)=\frac{G''(\xi_1)}{2}x^2$ ①

$G(x)=G(1)+G'(1)(1-x)+\frac{G''(\xi_2)}{2}(1-x)^2$，$\xi_2\in(x,1)$

$=G(1)+\frac{G''(\xi_2)}{2}(1-x)^2$ ②

① − ② 得 $0 = -G(1) + \frac{G''(\xi_1)}{2}x^2 - \frac{G''(\xi_2)}{2}(1-x)^2$

$G(1) = \frac{G''(\xi_1)}{2}x^2 - \frac{G''(\xi_2)}{2}(1-x)^2$

$|G(1)| = \frac{x^2}{2}|G''(\xi_1)| + \frac{(1-x)^2}{2}|G''(\xi_2)|$

$\le \frac{x^2+x^2-2x+1}{2}M$

$= \frac{2x^2-2x+1}{2}M$

令 $\frac{2x^2-2x+1}{2} = \frac{1}{4} \Rightarrow x = \frac{1}{2}$

设 $f(x)$ 在 $[0,1]$ 上有一阶连续导函数，且 $f(0)=f(1)=0$

求证：$\int_0^1 |f(x)|\,dx \le \frac{1}{4}\max\limits_{0\le x\le 1}|f'(x)|$ 　　$\int_0^1|f(x)|dx \ge \left|\int_0^1 f(x)dx\right|$ 比上题结论强

之前的解题流程是：令 $G(x)=\int_0^x f(t)dt$ 隐藏积分符号

但这里没法这样做，引出方法二：**先展开，后积分**

记 $M = \max\limits_{0\le x\le 1}|f'(x)|$

$f(x) = f(0) + f'(\xi_1)\cdot x, \quad \xi_1 \in (0,x)$

$\Leftrightarrow f(x) = f'(\xi_1)\cdot x$

$\Rightarrow |f(x)| = |f'(\xi_1)\cdot x| \le Mx$ 　　①

$f(x) = f(1) + f'(\xi_2)\cdot(x-1), \quad \xi_2 \in (x,1)$

$\Leftrightarrow f(x) = f'(\xi_2)\cdot(x-1)$

$\Rightarrow |f(x)| = |f'(\xi_2)|(1-x) \le M(1-x)$ 　　②

$|f(x)| \le \min\{Mx, M(1-x)\}$ （不等式再关系）

$\int_0^1 |f(x)|\,dx \le \int_0^1 \min\{Mx, M(1-x)\}\,dx$

$$\int_0^1 \min\{nx, n(1-x)\}dx = \left(\int_0^{\frac{1}{2}} + \int_{\frac{1}{2}}^1\right) \min\{nx, n(1-x)\} dx$$

$$= \int_0^{\frac{1}{2}} nx\,dx + \int_{\frac{1}{2}}^1 n(1-x)\,dx$$

$$= n\frac{1}{2}x^2\Big|_0^{\frac{1}{2}} + n\left(x-\frac{1}{2}x^2\right)\Big|_{\frac{1}{2}}^1 = \frac{1}{4}n$$

设 $f(x)$ 在 $[a,b]$ 上连续，且 $f(x)>0$，$f''(x)\le 0$，

证明：$\max\limits_{x\in[a,b]} f(x) \le \dfrac{2}{b-a}\int_a^b f(x)\,dx$

设 $f(c) = \max\limits_{x\in[a,b]} f(x)$，$c\in[a,b]$

(不能用费马定理，因为 c 可能在 a，b)

$$f(c) = f(x) + f'(x)\cdot(c-x) + \frac{f''(\xi)}{2}(c-x)^2,\ \xi \text{ 介于 } c \text{ 与 } x \text{ 之间}$$

$$\le f(x) + f'(x)\cdot(c-x)$$

$$\int_b^a f(c)\,dx \le \int_a^b f(x)\,dx + \int_a^b f'(x)(c-x)\,dx$$

$$(b-a)f(c) \le \int_a^b f(x)\,dx + (c-x)f(x)\Big|_a^b + \int_a^b f(x)\,dx$$

$$(b-a)f(c) \le 2\int_a^b f(x)\,dx + (c-b)f(b) - (c-a)f(a)$$

$$\le 2\int_a^b f(x)\,dx$$

$$\Leftrightarrow\quad f(c) \le \frac{2}{b-a}\int_a^b f(x)\,dx$$

(积分形式的琴生不等式)

$f(x)$ 在 $[a,b]$ 上连续，$\varphi''(x)\le 0$，

证明：$\dfrac{1}{b-a}\int_a^b \varphi(f(x))\,dx \le \varphi\left(\dfrac{1}{b-a}\int_a^b f(x)\,dx\right)$

记 $x_0 = \dfrac{1}{b-a}\int_a^b f(x)\,dx$

$$\varphi(x) = \varphi(x_0) + \varphi'(x_0)\cdot(x-x_0) + \frac{\varphi''(\xi)}{2}(x-x_0)^2,\ \xi \text{ 介于 } x_0 \text{ 与 } x \text{ 之间}$$

$$\leq \varphi(x_0) + \varphi'(x_0)\cdot(x - x_0)$$

$$\varphi(f(x)) \leq \varphi(x_0) + \varphi'(x_0)(f(x) - x_0) = \varphi(x_0) + \varphi'(x_0)\cdot f(x) - \varphi'(x_0)\cdot x_0$$

$$\int_a^b \varphi(f(x))\,dx \leq (b-a)\varphi(x_0) + \varphi'(x_0)\left(\int_a^b f(x)\,dx - \int_a^b x_0\,dx\right)$$

$$= (b-a)\varphi(x_0) + \varphi'(x_0)\left[\int_a^b f(x)\,dx - \frac{b-a}{b-a}\int_a^b f(x)\,dx\right]$$

$$= (b-a)\varphi(x_0)$$

$$\Rightarrow \frac{1}{b-a}\int_a^b \varphi(f(x))\,dx \leq \varphi(x_0) = \varphi\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)$$

积分第一中值定理

若 $f(x)$ 在 $[a,b]$ 上连续，$g(x)$ 在 $[a,b]$ 上可积，且 $g(x)$ 在 $[a,b]$ 上不变号，

则 $\exists \xi\in[a,b]$，使得 $\int_a^b f(x)g(x)\,dx = f(\xi)\cdot\int_a^b f(x)\,dx$

$m=\min\limits_{[a,b]} f(x) \qquad M=\max\limits_{[a,b]} f(x)$

不妨设 $g(x)\geqslant 0$

$m \leqslant f(x) \leqslant M$

$\Longleftrightarrow \quad m g(x) \leqslant f(x)\cdot g(x) \leqslant M g(x)$

$\Rightarrow \quad \int_a^b m g(x)dx = \int_a^b f(x)g(x)dx = \int_a^b M g(x)dx$

$\Rightarrow \quad m\int_a^b g(x)\,dx = \int_a^b f(x)g(x)\,dx = M\int_a^b g(x)dx$

① $\int_a^b g(x)\,dx = 0$ ，$\int_a^b f(x)g(x)\,dx = 0$

$\Rightarrow \exists \xi\in[a,b]$ 使得 $\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)dx$

② $\int_a^b g(x)\,dx \neq 0$，$g(x)\geqslant 0 \Rightarrow \int_a^b g(x)\,dx > 0$

$$m \leqslant \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \leqslant M$$

（介值定理）$\exists \xi\in[a,b]$，$f(\xi)=\dfrac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)\,dx}$

设 $f(x)$ 在 $[a,b]$ 上具有连续的导数，

证明：$\left|\frac{1}{b-a}\int_a^b f(x)\,dx\right| + \int_a^b |f'(x)|\,dx \geqslant \max\limits_{x\in[a,b]}|f(x)|$

（闭区间积分中值定理）$\exists \xi\in[a,b]$ $f(\xi)=\dfrac{\int_a^b f(x)dx}{b-a}$

设 $|f(c)|=\max\limits_{[a,b]}|f(x)|$，$c\in[a,b]$

即证 $\int_a^b |f'(x)|\,dx \geqslant |f(c)| - |f(\xi)|$

(三角不等式) $|f(c)| - |f(\xi)| \le |f(c) - f(\xi)| = |\int_\xi^c f'(x)dx|$

$\le \int_{\min\{c,\xi\}}^{\max\{c,\xi\}} |f'(x)| dx \le \int_a^b |f'(x)| dx$

设 $f(x)$ 在 $[0,1]$ 上连续可导，证明：

$\int_0^1 |f(x)| dx \le \max\{\int_0^1 |f'(x)| dx, |\int_0^1 f(x) dx|\}$

记 $A = \int_0^1 |f(x)| dx$ ， $B = \int_0^1 |f'(x)| dx$ ， $C = |\int_0^1 f(x) dx|$

(三角不等式) $\int_0^1 |f(x)| dx \ge |\int_0^1 f(x) dx|$ 即 $A \ge C$

① $A = C$ ， $A = C \le \max\{B, C\}$

② $A > C$ ， 要证 $A \le \max\{B, C\}$ ， 只需证 $A \le B$

即证 $\int_0^1 |f(x)| dx \le \int_0^1 |f'(x)| dx$

$A > C \Leftrightarrow \int_0^1 |f(x)| dx > |\int_0^1 f(x) dx| \Leftrightarrow f(x)$ 在 $(0,1)$ 上有正有负

$\Rightarrow \exists \delta \in (0,1)$ ， $f(\delta) = 0$

(闭区间积分中值定理) $\exists \xi \in [0,1]$ ， $\int_0^1 |f(x)| dx = |f(\xi)|$

$|f(\xi)| = |f(\xi) - f(\delta)| = |\int_m^n f'(x) dx| \le \int_m^n |f'(x)| dx \le \int_0^1 |f'(x)| dx$

其中 $m = \min\{\xi, \delta\}$ ， $n = \max\{\xi, \delta\}$

设 $f(x)$ 在 $[a,b]$ 上连续，且单调递增，

证明： $\int_a^b x f(x) dx \ge \frac{a+b}{2} \int_a^b f(x) dx$

即证： $\int_a^b (x - \frac{a+b}{2}) f(x) dx \ge 0$

$\int_a^b (x - \frac{a+b}{2}) f(x) dx = (\int_a^{\frac{a+b}{2}} + \int_{\frac{a+b}{2}}^b)(x - \frac{a+b}{2}) f(x) dx$

$$= \int_a^{\frac{a+b}{2}} \left(x - \frac{a+b}{2}\right) f(x)\,dx + \int_{\frac{a+b}{2}}^{b} \left(x - \frac{a+b}{2}\right) f(x)\,dx$$

（第一积分中值） $\exists\, \delta_1 \in \left[a, \frac{a+b}{2}\right]$,

$$\int_a^{\frac{a+b}{2}} \left(x - \frac{a+b}{2}\right) f(x)\,dx = f(\delta_1) \int_a^{\frac{a+b}{2}} \left(x - \frac{a+b}{2}\right) dx$$

$$= -\frac{(b-a)^2}{8} f(\delta_1)$$

$\exists\, \delta_2 \in \left[\frac{a+b}{2}, b\right]$

$$\int_a^{\frac{a+b}{2}} \left(x - \frac{a+b}{2}\right) f(x)\,dx = f(\delta_2) \cdot \int_a^{\frac{a+b}{2}} \left(x - \frac{a+b}{2}\right) dx$$

$$= \frac{(b-a)^2}{8} f(\delta_2)$$

$$\int_a^b \left(x - \frac{a+b}{2}\right) f(x)\,dx = \frac{(b-a)^2}{8} \left[f(\delta_2) - f(\delta_1)\right]$$

又 $f(x) \nearrow$, $a \le \delta_1 \le \frac{a+b}{2} \le \delta_2 \le b \Rightarrow f(\delta_2) > f(\delta_1)$

（另解）

$$\int_a^{\frac{a+b}{2}} \left(x - \frac{a+b}{2}\right) \cdot f(x)\,dx + \int_{\frac{a+b}{2}}^{b} \left(x - \frac{a+b}{2}\right) f(x)\,dx$$

令 $t = a+b-x$, $x: \frac{a+b}{2} \to b$, $t: \frac{a+b}{2} \to a$, $x = a+b-t$

$$\int_{\frac{a+b}{2}}^{b} \left(x - \frac{a+b}{2}\right) f(x)\,dt = \int_{\frac{a+b}{2}}^{a} \left(\frac{a+b}{2} - t\right) f(a+b-t)\,d(a+b-t)$$

$$= \int_a^{\frac{a+b}{2}} \left(t - \frac{a+b}{2}\right) f(a+b-t)\,d(a+b-t)$$

$$= \int_a^{\frac{a+b}{2}} -\left(t - \frac{a+b}{2}\right) f(a+b-t)\,dt$$

（换名） $= \int_a^{\frac{a+b}{2}} -\left(x - \frac{a+b}{2}\right) f(a+b-x)\,dx$

$$\int_a^{\frac{a+b}{2}} \left(x - \frac{a+b}{2}\right) \cdot f(x)\,dx + \int_{\frac{a+b}{2}}^{b} \left(x - \frac{a+b}{2}\right) f(x)\,dx$$

$$= \int_a^{\frac{a+b}{2}} \left(x - \frac{a+b}{2}\right) \left[f(x) - f(a+b-x)\right] dx$$

$x \in [a, \frac{a+b}{2}] \qquad (x - \frac{a+b}{2}) < 0$

$a \leq x \leq \frac{a+b}{2} \leq a+b-x \leq b \Rightarrow f(x) \leq f(a+b-x)$

$\Rightarrow \int_a^{\frac{a+b}{2}} (x-\frac{a+b}{2})[f(x)-f(a+b-x)]dx \geqslant 0$

(另解，利用单调性构造函数)

即证：$\int_a^b xf(x)dx - \frac{a+b}{2}\int_a^b f(x)dx \geqslant 0$

把 b 换成 t

设 $G(t) = \int_a^t xf(x)dx - \frac{a+t}{2}\int_a^t f(x)dx$

$G'(t) = tf(t) - \frac{1}{2}\int_a^t f(x)dx - \frac{a+t}{2}f(t)$

$= \frac{t-a}{2}f(t) - \frac{1}{2}\int_a^t f(x)dx$

$x \in [a,t] \Rightarrow x \leq t, \quad f(x)\nearrow \Rightarrow f(x) \leq f(t)$

$\Rightarrow \int_a^t f(x)dx \leq \int_a^t f(t)dx = f(t)(t-a)$

$G'(t) = \frac{t-a}{2}f(t) - \frac{1}{2}\int_a^t f(x)dx \geqslant \frac{t-a}{2}f(t) - \frac{t-a}{2}f(t) = 0$

$\Rightarrow G(t) \nearrow \Rightarrow G(b) \geqslant G(a) = 0$

设 $f(x)$ 在 $[0,1]$ 是非负的、单调递减的连续函数，且 $0<a<b<1$

证明：$\int_0^a f(x)dx \geqslant \frac{a}{b}\int_a^b f(x)dx$

即证：$\int_0^a f(x)dx - \frac{a}{b}\int_a^b f(x)d \geqslant 0 \Leftrightarrow b\int_0^a f(x)dx - a\int_a^b f(x)dx \geqslant 0$

设 $G(t) = t\int_0^a f(x)dx - a\int_a^t f(x)dx$

$G(a) = a\int_0^a f(x)dx, \qquad f(x) \geqslant 0 \Rightarrow \int_0^a f(x)dx \geqslant 0 \Rightarrow G(a) \geqslant 0$

$G'(t) = \int_0^a f(x)dx - af(t)$

$x \in [0,a], \quad f(x)\downarrow \Rightarrow f(x) \geqslant f(a)$

$\Rightarrow \int_0^a f(x)dx \geqslant \int_0^a f(a)dx = af(a)$

$G'(a) \geqslant af(a) - af(t)$

$t \geqslant a$, $f(x) \downarrow$, $f(t) \leqslant f(a)$

$\Rightarrow G'(a) \geqslant af(a) - af(t) \geqslant 0 \Rightarrow G(t) \nearrow$

从而 $G(b) \geqslant G(a) \geqslant 0$

(另解，用 a 构造)

$G(t) = \int_0^t f(x)dx - \frac{t}{b}\int_t^b f(x)dx$, $t \in [0,a]$, $G(0)=0$

$G'(t) = f(t) - \frac{1}{b}\int_t^b f(x)dx - \frac{t}{b}\cdot(-f(t))$

$= (1+\frac{t}{b})f(t) - \frac{1}{b}\int_t^b f(x)dx$

$x \in [t,b]$, $f(x)\downarrow \Rightarrow f(t) \geqslant f(x)$

$\Rightarrow \int_t^b f(x)dx \leqslant \int_t^b f(t)dx = (b-t)f(t)$

$G'(t) = (1+\frac{t}{b})f(t) - \frac{1}{b}\int_t^b f(x)dx \geqslant (1+\frac{t}{b})f(t) - \frac{b-t}{b}f(t)$

$= \frac{2t}{b}f(t)$

$f(t) \geqslant 0$, $t \in [0,a] \Rightarrow tf(t) \geqslant 0$

$G'(t) \geqslant \frac{2t}{b}f(t) \geqslant 0 \Rightarrow G(t) \nearrow$

$\Rightarrow G(a) \geqslant G(0)$

(另解，积分中值定理)

(闭区间积分中值) $\exists \xi_1 \in [0,a]$, $af(\xi_1) = \int_0^a f(x)dx$

$\exists \xi_2 \in [a,b]$, $(b-a)f(\xi_2) = \int_a^b f(x)dx$

即证 $af(\xi_1) \geqslant \frac{a}{b}(b-a)f(\xi_2)$

$\frac{a}{b}\cdot(b-a)f(\xi_2) = af(\xi_2) - \frac{a^2}{b}f(\xi_2) \leqslant af(\xi_2)$ $(f(x) \geqslant 0)$

$0 \leqslant \xi_1 \leqslant a \leqslant \xi_2 \leqslant b$, $f(x)\downarrow \Rightarrow f(\xi_1) \geqslant f(\xi_2)$

$\Rightarrow af(\xi_1) \geqslant af(\xi_2) \geqslant \frac{a}{b}(b-a)f(\xi_2)$

设 $f(x)$ 在 $[0,+\infty)$ 是单调递减的连续函数

证明：当 $a \geq 0$ 时，有 $\int_0^a (a^2-3x^2) f(x)\,dx \geq 0$

令 $G(t) = \int_0^t (t^2-3x^2) f(x)\,dx$

$= t^2 \int_0^t f(x)\,dx - \int_0^t 3x^2 f(x)\,dx$

$t \in [0,+\infty)$, $\quad G(0)=0$

$G'(t) = 2t\int_0^t f(x)\,dx + t^2 f(t) - 3t^2 f(t)$

$= 2t\int_0^t f(x)\,dx - 2t^2 f(t)$

$= 2t\left[\int_0^t f(x)\,dx - t f(t)\right]$

$x \in [0,t]$, $f(x)\downarrow \Rightarrow f(x) \geq f(t)$

$\Rightarrow \int_0^t f(x)\,dx \geq \int_0^t f(t)\,dt = t f(t)$

$\Rightarrow \int_0^t f(x)\,dx - t f(t) \geq 0$

$\Rightarrow G'(t) = 2t\cdot\left[\int_0^t f(x)\,dx - t f(t)\right] \geq 0$

$\Rightarrow G(t) \nearrow \Rightarrow G(a) \geq G(0) = 0$

积分与不等式 > 利用二重积分

如果积分区域 D 关于 $y=x$ 这条直线对称，

则 $\iint_D f(x,y)\,dxdy = \iint_D f(y,x)\,dxdy$

设 $f(x)$ 在 $[a,b]$ 上连续，$f(x) \geq 0$，$\int_a^b f(x)\,dx = 1$，$k \in R$，

证明：$\left(\int_a^b f(x)\cos kx\,dx\right)^2 + \left(\int_a^b f(x)\sin kx\,dx\right)^2 \leq 1$

$\left(\int_a^b f(x)\cos kx\,dx\right)^2 = \left(\int_a^b f(x)\cos kx\,dx\right)\left(\int_a^b f(x)\cos kx\,dx\right)$

$= \left(\int_a^b f(x)\cdot\cos kx\,dx\right)\left(\int_a^b f(y)\cdot\cos ky\,dy\right)$

$= \iint_D f(x)\cdot f(y)\cdot\cos kx\cdot\cos ky\,dxdy$ 其中 $D=\{(x,y)\mid a\leq x\leq b,\ a\leq y\leq b\}$

同理 $\left(\int_a^b f(x)\cdot\sin kx\,dx\right)^2 = \iint_D f(x)\cdot f(y)\cdot\sin kx\cdot\sin ky\,dxdy$

$\left(\int_a^b f(x)\cos kx\,dx\right)^2 + \left(\int_a^b f(x)\cdot\sin kx\,dx\right)^2$

$= \iint_D f(x)\cdot f(y)\cdot(\cos kx\cos ky + \sin kx\sin ky)\,dxdy$

$= \iint_D f(x)\cdot f(y)\cdot\cos(kx-ky)\,dxdy$

$\cos(kx-ky) \leq 1$

$f(x)\geq 0 \Rightarrow f(x)\cdot f(y)\cdot\cos(kx-ky) \leq f(x)\cdot f(y)$

$\Rightarrow \iint_D f(x)\cdot f(y)\cdot\cos(kx-ky)\,dxdy \leq \iint_D f(x)\cdot f(y)\,dxdy$

$= \int_a^b dy\int_a^b f(x)\cdot f(y)\cdot dx = \int_a^b f(y)\,dy\cdot\int_a^b f(x)\cdot dx = 1$

设函数 $f(x) \in [a,b]$，不恒为 0，满足 $0 \leq f(x) \leq M$

证明：$\left(\int_a^b f(x)\cdot\cos x\right)^2 + \left(\int_a^b f(x)\cdot\sin x\right)^2 + \frac{M^2(b-a)^4}{12} \geq \left(\int_a^b f(x)dx\right)^2$

$\left(\int_a^b f(x)\cdot\cos x dx\right)^2 = \iint_D f(x)\cdot f(y)\cdot\cos x\cdot\cos y\, dxdy$，$D=\left\{(x,y)\,\middle|\,\begin{array}{l}a\leq x\leq b\\ a\leq y\leq b\end{array}\right\}$

$\left(\int_a^b f(x)\cdot\sin x dx\right)^2 = \iint_D f(x)\cdot f(y)\cdot\sin x\cdot\sin y\, dxdy$

$\left(\int_a^b f(x)\cdot dx\right)^2 = \iint_D f(x)\cdot f(y)\, dxdy$

即证：$\iint_D f(x)\cdot f(y)\cdot[1-\cos x\cos y-\sin x\sin y]\,dxdy \leq \frac{M^2(b-a)^4}{12}$

$1-\cos x\cos y-\sin x\sin y = 1-(\cos x\cos y+\sin x\sin y) = 1-\cos(x-y)$

$= 1-\left(1-2\sin^2\frac{x-y}{2}\right) = 2\sin^2\frac{x-y}{2} \leq 2\left(\frac{x-y}{2}\right)^2$ $(|\sin x|<|x|)$

$= \frac{(x-y)^2}{2}$

$\iint_D f(x)\cdot f(y)\cdot\frac{(x-y)^2}{2}dxdy \leq \iint_D M^2\frac{(x-y)^2}{2}dxdy$

$= \frac{M^2}{2}\int_a^b dy\int_a^b (x-y)^2 dx = \frac{M^2}{2}\int_a^b \frac{1}{3}(x-y)^3\Big|_{x=a}^{x=b} dy$

$= \frac{M^2}{6}\int_a^b (b-y)^3-(a-y)^3 dy = \frac{M^2}{6}\int_a^b (y-a)^3 dy - \frac{M^2}{6}\int_a^b (y-b)^3 dy$

$= \frac{M^2}{6}\cdot\frac{1}{4}(y-a)^4\Big|_a^b - \frac{M^2}{6}\cdot\frac{1}{4}(y-b)^4\Big|_a^b$

$= \frac{M^2}{24}\left[(b-a)^4+(b-a)^4\right] = \frac{M^2}{12}(b-a)^4$ 得证

$f(x), g(x)$ 在 $[a,b]$ 上连续，且 $f'(x)$ 在 $[a,b]$ 上 $\nearrow$，$g(x)$ 在 $[a,b]$ 上 $\searrow$

证明：$(b-a)\int_a^b f(x)g(x)dx \leq \left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right)$

$(b-a)\int_a^b f(x)\cdot g(x)dx = \int_a^b 1dx\int_a^b f(x)\cdot g(x)dx = \iint_D f(y)\cdot g(y)\,dxdy$

$\left(\int_a^b f(x)dx\right)\left(\int_a^b g(x)dx\right) = \iint_D f(x)\cdot g(y)\,dxdy$

其中 $D=\{(x,y)\mid a\leq x\leq b,\ a\leq y\leq b\}$

即证 $\iint_D f(y)\cdot g(y)\,dxdy \leq \iint_D f(x)\cdot g(y)\,dxdy$

即证 $\iint_D [f(x)-f(y)]\cdot g(y)\,dxdy \geqslant 0$ ☆

$\iint_D [f(x)-f(y)]\cdot g(y)\,dxdy = \iint_D [f(y)-f(x)]\cdot g(x)\,dxdy$

（D 关于 $y=x$ 对称）☆

$\iint_D [f(x)-f(y)]\cdot g(y)\,dxdy = \frac{1}{2}\left[\iint_D (f(x)-f(y))\cdot g(y)\,dxdy + \iint_D (f(y)-f(x))\cdot g(x)\,dxdy\right]$

$= \frac{1}{2}\iint_D (f(x)-f(y))(g(y)-g(x))\,dxdy$

1) $x \geqslant y$，$f(x)\nearrow \Rightarrow f(x)\geqslant f(y)$

$g(x)\downarrow \Rightarrow g(x)\leqslant g(y)$

$(f(x)-f(y))\cdot(g(y)-g(x)) \geqslant 0$

2) $x<y$，$f(x)\nearrow \Rightarrow f(x)\leqslant f(y)$

$g(x)\downarrow \Rightarrow g(x)\geqslant g(y)$

$(f(x)-f(y))\cdot(g(y)-g(x)) \geqslant 0$

从而 $(f(x)-f(y))\cdot(g(y)-g(x)) \geqslant 0$ 在 D 上恒成立

从而 $\iint_D (f(x)-f(y))\cdot(g(y)-g(x))\,dxdy \geqslant 0$

柯西不等式 $(a^2+b^2)(x^2+y^2) \geqslant (ax+by)^2$

$f(x)$，$g(x)$ 在 $[a,b]$ 上可积，（积分形式的柯西不等式）

求证：$\left(\int_a^b f(x)\cdot g(x)\,dx\right)^2 \leqslant \left(\int_a^b f^2(x)\,dx\right)\left(\int_a^b g^2(x)\,dx\right)$

$\left(\int_a^b f(x)\cdot g(x)\,dx\right)^2 = \iint_D f(x)\cdot g(x)\cdot f(y)\cdot g(y)\,dxdy$

$\left(\int_a^b f^2(x)\,dx\right)^2\left(\int_a^b g(x)\,dx\right)^2 = \iint_D f^2(x)\,g^2(y)\,dxdy$

其中 $D=\{(x,y) \mid a\leqslant x\leqslant b,\ a\leqslant y\leqslant b\}$

即证：$\iint_D f^2(x)g^2(y) - f(x)\cdot f(y)\cdot g(x)\cdot g(y)\,dxdy \geqslant 0$

$\iint_D f^2(x)\cdot g^2(y) - f(x)\cdot f(y)\cdot g(x)\cdot g(y) \, dxdy$

$= \iint_D f(x)\cdot (f(x)\cdot g^2(y) - f(y)\cdot g(x)\cdot g(y)) \, dxdy$

$= \iint_D f(x)\cdot g(y) (f(x)\cdot g(y) - f(y)\cdot g(x)) \, dxdy$

D 关于 $y=x$ 对称

$\iint_D f(x)\cdot g(y) (f(x)\cdot g(y) - f(y)\cdot g(x)) \, dxdy$

$= \iint_D f^2(y)\cdot g^2(x) - f(x)\cdot f(y)\cdot g(x)\cdot g(y)) \, dxdy$

$= \iint_D f(y)\cdot g(x)\cdot (f(y)\cdot g(x) - f(x)\cdot g(y)) \, dxdy$

$= \frac{1}{2}\iint_D [f(x)\cdot g(y) - f(y)\cdot g(x)][f(x)\cdot g(y) - f(y)\cdot g(x)] \, dxdy$

$= \frac{1}{2}\iint_D [f(x)\cdot g(y) - f(y)\cdot g(x)]^2 \, dxdy \geq 0$

(另解，构造抛物线 辅助函数)

(分析) $b=\int_0^1 f(x)g(x)dx$, $a=\int_0^1 f^2(x)dx$ $c=\int_0^1 g^2(x)dx$

$g(t) = \int_0^1 f^2(x)dx\cdot t^2 - 2\int_0^1 f(x)g(x)dx\cdot t + \int_0^1 g^2(x)dx$

$= \int_0^1 f^2(x)\cdot t^2 - 2f(x)g(x) + g^2(x) \, dx$

$= \int_0^1 (f(x)t - g(x))^2 \, dx$

(开始做题)

$g(t) = \int_0^1 f^2(x)dx\cdot t^2 - 2\int_0^1 f(x)g(x)dx\cdot t + \int_0^1 g^2(x)dx$

$= \int_0^1 f^2(x)\cdot t^2 - 2f(x)g(x)\cdot t + g^2(x)dx$

$= \int_0^1 (f(x)\cdot t - g(x))^2 dx$

$(f(x)\cdot t - g(x))^2 \geq 0$

$\Rightarrow g(t) = \int_0^1 (f(x)\cdot t - g(x))^2 dx \geq 0$

$\Rightarrow \Delta = (2\int_0^1 f(x)g(x))^2 - 4\int_0^1 f^2(x)dx\cdot\int_0^1 g^2(x)dx \leq 0$

$\Rightarrow (\int_0^1 f(x)g(x)dx)^2 \leq (\int_0^1 f^2(x)dx)(\int_0^1 g^2(x)dx)$

积分与不等式 > 对简单不等式积分

从条件中发现简单的不等式，然后对其积分

设函数 $f(x)$ 在 $[-\frac{1}{a}, a]$ 可积，$f(x) \geq 0$，且 $\int_{-\frac{1}{a}}^{a} x f(x)\,dx = 0$

证明：$\int_{-\frac{1}{a}}^{a} x^2 f(x)\,dx \leq \int_{-\frac{1}{a}}^{a} f(x)\,dx$ $(a > 0)$

(分析) 即证 $\int_{-\frac{1}{a}}^{a} (x^2-1) f(x)\,dx \leq 0$

$\int_{-\frac{1}{a}}^{a} x f(x)\,dx = 0 \Rightarrow \int_{-\frac{1}{a}}^{a} (x^2-1) f(x)\,dx = \int_{-\frac{1}{a}}^{a} (x^2-1) f(x)\,dx + K\int_{-\frac{1}{a}}^{a} x f(x)\,dx$

(待定 K) $= \int_{-\frac{1}{a}}^{a} (x^2+Kx-1) f(x)\,dx$

记 $g(x) = x^2+Kx-1$ 开口向上

$\int_{-\frac{1}{a}}^{a} (x^2-1) f(x)\,dx = \int_{-\frac{1}{a}}^{a} g(x)\cdot f(x)\,dx$

要证 $\int_{-\frac{1}{a}}^{a} g(x) f(x)\,dx \leq 0$，只需证 $\exists K$，使得对 $\forall x \in (-\frac{1}{a}, a)$，$g(x) \leq 0$

取 $K = (\frac{1}{a} - a)$ 时，有 $g(x) = x^2 + (\frac{1}{a} - a)x - 1 = (x+\frac{1}{a})(x-a) \leq 0$

对 $\forall x \in (-\frac{1}{a}, a)$ 成立

(开始做题)

$\forall x \in [-\frac{1}{a}, a]$，$x^2 + (\frac{1}{a} - a)x - 1 \leq 0$

$\Rightarrow (x^2 + (\frac{1}{a} - a)x - 1) f(x) \leq 0$ $(f(x) \geq 0)$

$\Rightarrow \int_{-\frac{1}{a}}^{a} (x^2 + (\frac{1}{a} - a)x - 1) f(x)\,dx \leq 0$

$\Rightarrow \int_{-\frac{1}{a}}^{a} (x^2-1) f(x)\,dx + (\frac{1}{a} - a)\int_{-\frac{1}{a}}^{a} x f(x)\,dx \leq 0$

$\Rightarrow \int_{-\frac{1}{a}}^{a} (x^2-1) f(x)\,dx \leq 0$ $(\int_{-\frac{1}{a}}^{a} x f(x)\,dx = 0)$

$\Rightarrow \int_{-\frac{1}{a}}^{a} x^2 f(x)\,dx \leq \int_{-\frac{1}{a}}^{a} f(x)\,dx$

设 $f(x)$ 在 $[0,1]$ 上连续，且 $1\le f(x)\le 3$，

证明：$\int_0^1 f(x)dx \int_0^1 \frac{1}{f(x)}dx \le \frac{4}{3}$

$1\le f(x)\le 3 \Rightarrow f(x)-3\le 0,\quad f(x)-1\ge 0$

当3产生 $\frac{1}{f}$

$1\le f(x)\le 3 \Rightarrow \frac{1}{3}\le \frac{1}{f(x)}\le 1 \Rightarrow \frac{1}{f(x)}-\frac{1}{3}\ge 0,\ \frac{1}{f(x)}-1\le 0$

$(f-1)(\frac{1}{f}-\frac{1}{3}) = 1-\frac{1}{3}f-\frac{1}{f}+\frac{1}{3}\ge 0$

$\Rightarrow \frac{4}{3}\ge \frac{1}{3}f+\frac{1}{f}$

$\Rightarrow \int_0^1 \frac{1}{3}f(x)+\frac{1}{f(x)}dx \le \int_0^1 \frac{4}{3}dx = \frac{4}{3}$

$\int_0^1 \frac{1}{3}f(x)+\frac{1}{f(x)}dx = \int_0^1 \frac{1}{3}f(x)dx + \int_0^1 \frac{1}{f(x)}dx$

$\ge 2\sqrt{\int_0^1 \frac{1}{3}f(x)dx\cdot\int_0^1 \frac{1}{f(x)}dx}$

$\frac{4}{3}\ge \int_0^1 \frac{1}{3}f(x)+\frac{1}{f(x)}dx \ge 2\sqrt{\int_0^1 \frac{1}{3}f(x)dx\cdot\int_0^1 \frac{1}{f(x)}dx}$

$\Rightarrow \frac{4}{9}\ge \frac{1}{3}\cdot\int_0^1 f(x)dx\int_0^1 \frac{1}{f(x)}dx$

$\Rightarrow \frac{4}{3}\ge \int_0^1 f(x)dx\cdot\int_0^1 \frac{1}{f(x)}dx$

（另解，构造辅助抛物线）

$g(t)=at^2-2bt+c$ 至少有一个零点 $\Leftrightarrow \Delta\le 0 \Rightarrow \exists t_0, g(t_0)\le 0$

$\Delta=(-2b)^2-4ac=4b^2-4ac\ge 0 \Leftrightarrow b^2\ge ac$

(分析) $a=\int_0^1 f(x)dx\quad c=\int_0^1 \frac{1}{f(x)}dx\quad b=\frac{2}{\sqrt{3}}$

$g(t)=\int_0^1 f(x)dx\cdot t^2-2\cdot\frac{2}{\sqrt{3}}t+\int_0^1\frac{1}{f(x)}dx$

$=\int_0^1 f(x)t^2dx-\int_0^1\frac{4}{\sqrt{3}}t\,dx+\int_0^1\frac{1}{f(x)}dx$

$=\int_0^1 f(x)\cdot t^2-\frac{4}{\sqrt{3}}t+\frac{1}{f(x)}dx$

$=\int_0^1 \frac{f^2(x)t^2-\frac{4}{\sqrt{3}}t\cdot f(x)+1}{f(x)}dx$

$b^2\ge ac \Leftrightarrow g(t)$ 至少存在一个零点 $\Leftrightarrow \exists t_0, g(t_0)\le 0$

（下面需要反解出 t_0）

$$g(t_0)=\int_0^1 \frac{f^2(x)t_0^2-\frac{4}{\sqrt3}t_0f(x)+1}{f(x)}dx \leq 0$$

$(1\leq f(x)\leq 3 \Rightarrow f(x)>0)$

$$\Leftrightarrow \int_0^1 f^2(x)t_0^2-\frac{4}{\sqrt3}t_0f(x)+1\,dx\leq 0$$

$$\Leftarrow f^2(x)\cdot t_0^2-\frac{4}{\sqrt3}t_0f(x)+1\leq 0$$

$$\Leftrightarrow t_0^2\left(f(x)-\frac{\sqrt3}{t_0}\right)\left(f(x)-\frac{1}{\sqrt3}\cdot\frac{1}{t_0}\right)\leq 0$$

$$\Leftarrow \frac{1}{\sqrt3}\cdot\frac{1}{t_0}\leq f(x)\leq \sqrt3\cdot\frac{1}{t_0} \quad (t_0>0)$$

$$\Leftarrow t_0=\frac{1}{\sqrt3}$$

（开始做题）

令 $g(t)=\int_0^1 f(x)\,dx\cdot t^2-2\cdot\frac{2}{\sqrt3}t+\int_0^1\frac{1}{f(x)}dx$

$$=\int_0^1\frac{f(x)t^2-\frac{4}{\sqrt3}tf(x)+1}{f(x)}dx$$

$$g\left(\frac{1}{\sqrt3}\right)=\int_0^1\frac{\frac{1}{3}f(x)-\frac{4}{3}f(x)+1}{f(x)}dx$$

$$=\int_0^1\frac{1-f(x)}{f(x)}dx$$

$$1\leq f(x)\leq 3 \Rightarrow \frac{1-f(x)}{f(x)}\leq 3$$

$$\Rightarrow g\left(\frac{1}{\sqrt3}\right)\leq 0$$

$1\leq f(x)\leq 3 \Rightarrow f(x)>0 \Rightarrow \int_0^1 f(x)\,dx>0 \Rightarrow g(t)$ 开口向上

$$\Delta=\left(\frac{4}{\sqrt3}\right)^2-4\int_0^1 f(x)dx\cdot\int_0^1\frac{1}{f(x)}dx\geqslant 0$$

$$\Leftrightarrow \frac{4}{3}-\int_0^1 f(x)dx\cdot\int_0^1\frac{1}{f(x)}dx\geqslant 0$$

$$\Leftrightarrow \int_0^1 f(x)\,dx\cdot\int_0^1\frac{1}{f(x)}dx\leq\frac{4}{3}$$ 得证